Position-Wavenumber Approximation for Propagation in a Random Dispersive Channel

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I. Introduction and Background

In an ocean channel, random fluctuations in sound speed, and frequency-dependent effects such as attenuation and dispersion, impact sound propagation and give rise to nonstationarities that can be detrimental to detection and classification. To understand, quantify and potentially mitigate such propagation effects, we develop a joint position-wavenumber approach for wave propagation in a random dispersive channel. We derive an approximation of the position-wavenumber (x-k) Wigner spectrum of the random channel, and its impact on a propagating acoustic wave. For simplicity we consider one spatial dimension, but the method can be readily extended to multiple dimensions.

Let $u(x,0)$ be an initial (deterministic) pulse, propagating in a random channel with Wigner spectrum $\overline{W}_h(x,k;\,t)$. The propagating wave $u(x,t)$ is therefore also random, and its Wigner spectrum $\overline{W}_u(x,k;\,t)$ is given by [1], [2]

$$\overline{W}_u(x,k;\,t) \;=\; E\left\{\frac{1}{2\pi}\int u^*\left(x-\frac{\lambda}{2},t\right)u\left(x+\frac{\lambda}{2},t\right)\,e^{-jk\lambda}\,d\lambda\right\} \;=\; \int W_u(x-x',k;\,0)\,\overline{W}_h(x',k;\,t)\,dx' \tag{1}$$

where $W_u(x,k;\,0)$ is the Wigner distribution of $u(x,0)$ and $E\{.\}$ denotes the expected value.

II. Results

Our aim is to develop approximations of this exact result, for different channel characteristics, which facilitate our understanding of the propagation effects. We have previously considered the deterministic case [3]. The random case raises additional challenges and considerations, and there are a greater number of possible approaches. One approach, which we present here, is to extend the deterministic case to the random case by introducing random parameters, and then ensemble averaging the deterministic approximation. We consider a dispersive channel with random sound speed c and exponential attenuation parameterized by the variable b, with joint pdf $P(b,c)$. For this case, the approximate Wigner spectrum of the channel is given by,

$$\overline{W}_h(x,k;\,t) \quad\approx\quad E\left\{e^{-bkt}\,\delta(x-v_g(k)\,t)\right\} \;=\; \frac{1}{|t\,v_1(k)|}\int e^{-bkt}\,P\left(b,\frac{x}{t\,v_1(k)}\right)\,db \tag{2}$$

where $v_g(k)=c\,v_1(k)$ is the group velocity, parameterized by random sound speed c and deterministic function $v_1(k)$. The dispersionless case corresponds to $v_1(k)=1$.

Example 1: Independent random damping and sound speed. Let the joint pdf $P(b,c)$ be independent, with sound speed c described by arbitrary distribution $P_c(c)$ and the damping parameter b given by an Erlang distribution, $P_b(b)=\frac{1}{(n-1)!}\,\lambda^n\,b^{n-1}e^{-\lambda b},\;\; b\geq 0$. Then, we have by (2) that the Wigner spectrum of the channel is approximately

$$\overline{W}_h(x,k;\,t) \quad\approx\quad \left(\frac{\lambda}{\lambda+kt}\right)^n\frac{1}{|t\,v_1(k)|}\,P_c\left(\frac{x}{t\,v_1(k)}\right) \tag{3}$$

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To obtain the approximate Wigner spectrum of the propagating wave, we convolve this channel spectrum with the Wigner spectrum of the initial wave, per Eq. (1). Note that as the wave evolves, i.e. as t increases, the channel increasingly attenuates the propagating wave, and the statistics of the wave in x are distributed according to $P_c\left(\frac{x}{t\,v_1(k)}\right)$.

If we take $P_c(c)$ to be Gaussian with mean c_0 and variance σ^2, then the approximation is

$$\overline{W}_h(x,k;\,t) \;\approx\; \left(\frac{\lambda}{\lambda+kt}\right)^n \frac{1}{\sqrt{2\pi\,t^2\,v_1^2(k)\,\sigma^2}}\,e^{-\frac{(x-c_0\,v_1(k)\,t)^2}{2\,t^2\,v_1^2(k)\,\sigma^2}} \tag{4}$$

Note that at $t=0$ the wave has not propagated, and the channel approximation becomes $\delta(x)$, which is a satisfying result in that Eq. (1) is exactly satisfied by the approximation at $t=0$. This is not true of other standard approximations, such as the stationary phase approximation which is not accurate for small times. Also note that even when there is no dispersion ($v_1(k)=1$), the channel still induces spreading on average in the wave, unlike the deterministic case for which the channel Wigner spectrum becomes $\delta(x-c\,t)$ in the absence of dispersion (and damping).

Example 2: Correlated damping and sound speed. In practice the damping and propagation velocity may be correlated. To illustrate the application of the Wigner approach for a pdf with correlations, consider the joint exponential density,

$$P(b,c) \;=\; (1+d)\lambda_b\,\lambda_c\,e^{-\lambda_b b-\lambda_c c} - 2d\lambda_b\lambda_c\left(e^{-2\lambda_b b-\lambda_c c}+e^{-\lambda_b b-2\lambda_c c}-2e^{-2\lambda_b b-2\lambda_c c}\right),\quad b,c\,\geq\,0 \tag{5}$$

where $-1\leq d\leq 1$ and $d=0$ corresponds to the correlationless case. The marginal distributions are $P_b(b) = \lambda_b\,e^{-\lambda_b b}$ and $P_c(c) = \lambda_c\,e^{-\lambda_c c}$. We consider this density because the Wigner spectrum is readily evaluated; by (2) we have

$$\overline{W}_h(x,k;\,t) \;\approx\; \frac{1}{|t\,v_1(k)|}\left[\frac{\lambda_b\,\lambda_c}{\lambda_b+kt}e^{-\lambda_c c}\left\{(1+d)\,-2de^{-\lambda_c c}\right\} - \frac{2d\lambda_b\lambda_c}{2\lambda_b+kt}e^{-\lambda_c c}\left\{1-2e^{-\lambda_c c}\right\}\right] \tag{6}$$

where $c=\frac{x}{t\,v_1(k)}$. Another pdf with different correlations but the same marginals is

$$P(b,c) \;=\; \lambda_b\,\lambda_c\,\left[(1+\theta\lambda_b b)(1+\theta\lambda_c c)-\theta\right]e^{-\lambda_b b-\lambda_c c-\theta\lambda_b\lambda_c bc},\qquad b,c\,\geq\,0 \tag{7}$$

for which we obtain

$$\overline{W}_h(x,k;\,t) \;\approx\; \frac{1}{|t\,v_1(k)|}\,\frac{\lambda_b\lambda_c}{\lambda_b(1+\theta\lambda_c c)+kt}e^{-\lambda_c c}\left[1-\theta+\theta\lambda_c c+\frac{\theta\lambda_b(1+\lambda_c c)}{\lambda_b(1+\theta\lambda_c c)+kt}\right] \tag{8}$$

where $c=\frac{x}{t\,v_1(k)}$, $0\leq\theta\leq 1$ and the correlations are negative, or zero (for $\theta=0$).

III. Conclusion

We have presented an approach based on the Wigner position-wavenumber distribution to characterize propagation in a dispersive random channel. This approach leads to a new approximation and gives insights to random wave propagation. Examples were presented to illustrate the method.

References

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